Summer 2025 DRP HW 6: Generators and Relations

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1 Notes

This first fall homework continues completes developing machinery we need to study Cayley graphs. The warmup will review some facts matrix groups and Cayley graphs. The bulk of this worksheet will involve describing groups using generators and relations. Some problems may also be duplicated in other texts. If this is the case, try to answer these without referring to the text. If you forget the statement of a theorem or formula, however, you are free to look up some results online.

2 Warmup: Matrix Groups

Let $GL_n(\mathbb{Z})$ and $SL_n(\mathbb{Z})$ be the group of invertible $n \times n$ (integer) matrices and $n \times n$ (integer) matrices of determinant 1, respectively.

Exercise 2.1. Let G be the subset of $SL_2(\mathbb{Z})$ consisting of all matrices of the form $A_n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$, for $n \in \mathbb{Z}$. Prove that G is a subgroup, and draw the Cayley graph of G with generators $\pm A_1$. [*Hint: Is G isomorphic to a more familiar group?*]

Exercise 2.2. Recall that $SL_2(\mathbb{Z})$ is generated by two matrices A and B satisfying $A^4 = (AB)^6 = I$. Use this information to draw the Cayley graph $\mathrm{Cay}(SL_2(\mathbb{Z}), \{\pm A, \pm B\})$ up to 4 units away from I (ie., draw all vertices and edges with distance at most 4 from the identity)

Recall that $\mathbb{Z}_p$ is the group of integers modulo p. For the following exercise, let p be a prime number. Let $SL_2(\mathbb{Z}_p)$ be the group of integral 2x2 matrices with entries in $\mathbb{Z}_p$.

Exercise 2.3. Consider the map $\varphi : SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}_p)$ given by taking mod p in each component. Prove that this is a homomorphism. Show that it is *not* an isomorphism, though.

Exercise 2.4. We will be concerned about the specific case $p = 2$. Firstly, note that there are precisely 4 matrices inside $SL_2(\mathbb{Z}_2)$. What are these matrices?

Using some combinatorics, prove that the map $\varphi : SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}_2)$ is surjective.[a] Let $\Gamma(2)$ be the kernel of this homomorphism. It is a normal subgroup of $SL_2(\mathbb{Z})$ [why?]. Prove that $\Gamma(2)$ is generated by the two matrices

$$S = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

[*Hint: start with some examples, and do exactly what you did to prove that $SL_2(\mathbb{Z})$ was generated by the matrices A and B.*]

Consider the matrices A and B generating $SL_2(\mathbb{Z})$, we had relations $A^4 = (AB)^6 = I$. A natural question would be if S and T satisfy similar (or any) relations in the group $\Gamma(2)$. It turns out that the answer to this question is no, and its proof is assigned in your reading in Meier.

[a] This is true for all primes p, but more complicated, and not as relevant to us.

3 Generators and Relations

This section presupposes that you've already read the assigned sections in Meier's *Groups, Graphs and Trees*. Let's start with a warmup problem involving ideas we have already worked with.

Exercise 3.1. Let G be the group with presentation $\langle A, B \mid A^4 = (AB)^6 = e\rangle$. Prove that G is isomorphic to $SL_2(\mathbb{Z})$ by considering the map $f : G \to SL_2(\mathbb{Z})$ given by

$$f(A) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad f(B) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Exercise 3.2. Denote F_2 as the free group generated by x and y and consider the map $\varphi : F_2 \to \mathbb{Z}^2$ given by $\varphi(x) = (1, 0)$ and $\varphi(y) = (0, 1)$. This is clearly a surjective map. Explicitly describe the kernel of this map. What does the first isomorphism theorem tell you?

This next and final exercise is of extreme importance in low dimensional topology; I use it daily in my research. Start with a finitely presented group $G = \langle x_1, \ldots, x_n \mid R_1 = R_2 = \cdots = R_k = 1\rangle$, and let $g_1, \ldots, g_m$ be some arbitrary group elements. Because we have declared a generating set, each g_j can be written as a *word* in the x_i's (eg., $g_3 = x_1 x_2^2 x_3^{-5} x_8$). Denote these words as $W_1, \ldots, W_k$. Now, let H be the normal subgroup generated by $\{g_1, \ldots, g_m\}$[1]

Exercise 3.3. Prove that the quotient group G/H has the presentation $G/H \cong \langle x_1, \ldots, x_n \mid R_1 = R_2 = \cdots = R_k = W_1 = \ldots W_k = 1\rangle$. That is, the quotient operation is seen as *adjoining a relation.*

If you believe the statement of this exercise, you may alleviate some of the abstract nonsense that comes from viewing quotient groups as coming from cosets and whatnot. In this view, a group is built

[1] Recall that the normal subgroup generated by some elements is the smallest normal subgroup containing the specified elements.

from generators (ie., letters) and certain relationships between the letters. The action of quotienting simply *adds a new relationship* to the existing list. To check your understanding, how does this exercise relate to Exercise 3.2? In particular, can you describe the kernel in the above question as the normal subgroup generated by some element and then apply Exercise 3.3?