Spring 2025 DRP HW 7: Differential Forms

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1 Notes

This homework dives into computations regarding differential forms. We have all implicitly seen this in our multivariable classes, so the calculations are not really any different; they are just packaged differently. Some of the results presented here will be of use in the future, and some are general cases of stronger results in smooth topology. Questions will vary in difficulty. Some problems may also be duplicated in other texts. If this is the case, try to answer these without referring to the text. If you forget the statement of a theorem or formula, however, you are free to look up some results online.

2 Warmup: The Pullback Revisited

Fair warning; this section is relatively abstract, and involves a lot of notation. I would recommend really sticking to the definitions and just performing the correct formal substitutions. Do not worry so much about intuition here - there isn't really any. This unit is mostly about developing a solid foundation in order to perform integration.

Recall from earlier in the course that a smooth function $f : \mathbb{R}^n \to \mathbb{R}^m$ induces a linear map $Jf(p) : \mathbb{R}^n \to \mathbb{R}^m$, where $Jf(p)$ is the Jacobian of f (a real $m \times n$ matrix where we plug in p wherever appropriate). Notice that there is a dependence on the point p; if we were to choose a different point p, we in general get a different linear map. In the context of tangent spaces, this is best summarised in the following.

Definition 2.1. Given a fixed point $p \in \mathbb{R}^n$ associate to $f : \mathbb{R}^n \to \mathbb{R}^m$ the linear map $f_* : \mathbb{R}^n_p \to \mathbb{R}^m_{f(p)}$ defined by $f_*(v_p) = [Jf(p)v]_{f(p)}$. f_* is called the **differential** of f.

Recall that the subscript tells us which vector spaces the vector lives. In this case, $Jf(p)v$ lives in $\mathbb{R}^m$, but $[Jf(p)v]_{f(p)}$ lives in $\mathbb{R}^m_{f(p)}$. Let us now move onto the next level of abstraction. Recall that any linear map $T : V \to W$ induces a **pullback** map $T^* : \Omega^k(W) \to \Omega^k(V)$. That is, given an alternating k-tensor $\varphi : W^k \to \mathbb{R}$, $T^*\varphi$ is the alternating k-tensor $T^*V : V^k \to \mathbb{R}$ given by

$$T^*\varphi(v_1, \dots v_k) = \varphi(Tv_1, \dots, Tv_k)$$

Let us do a concrete example to get some intuition for this construction.

Exercise 2.2. Recall that $\det(\cdot) \in \Omega^2(\mathbb{R}^2)$. Explicitly compute the pullback map $T^* \det \in \Omega^3(\mathbb{R}^3)$ under the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$

$$Tx = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 5 & 0 \end{bmatrix} x$$

On the other hand, let $S : \mathbb{R}^3 \to \mathbb{R}^4$. What is the pullback of any k-form $\varphi \in \Omega^4(\mathbb{R}^4)$?

We now combine the previous two ideas to reach our final level of abstraction. The differential $f_* : \mathbb{R}^n_p \to \mathbb{R}^m_{f(p)}$ is a linear map, therefore it admits a pullback $f^* : \Omega^k\left(\mathbb{R}^m_{f(p)}\right) \to \Omega^k(\mathbb{R}^n_p)$ for every k [*Notice that the asterisk is superscripted now, whereas before it was subscripted. Keep note of where sub/superscripts are as this section progesses*]. Now let ω be a k-form on $\mathbb{R}^m$. Recall that this means ω eats a point $x \in \mathbb{R}^m$ and returns a k-tensor $\omega(x) \in \Omega^k(\mathbb{R}^m_x)$. The pullback of the differential induces a k-form $f^*\omega$ on $\mathbb{R}^n$ given by

$$\underbrace{f^*\omega(p)}_{\in \Omega^k(\mathbb{R}^n_p)} = \underbrace{f^*(\underbrace{\omega(f(p))}_{\in \Omega^k\left(\mathbb{R}^m_{f(p)}\right)})}_{\in \Omega^k(\mathbb{R}^n_p)}$$

It is a matter of chasing definitions to explicitly write for $v_1, \ldots, v_k \in \mathbb{R}^n_p$

$$f^*\omega(p)(v_1, \ldots, v_k) = \omega(f(p))(f_*(v_1) \ldots, f_*(v_k))$$

There is admittedly a clutter of notation, but this is all necessary because there is straight up a lot of data to manage. Nevertheless, when we compute things, we rarely do so by definition - we have formulas to apply. As an example, suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^m \to \mathbb{R}$ and ω a k-form on $\mathbb{R}^m$. Then for $p \in \mathbb{R}^n$, we have (by definition) that if $v_1, \ldots, v_k \in \mathbb{R}^n_p$,

$$\begin{aligned} f^*(g \cdot \omega)(p)(v_1, \ldots, v_k) &= (g \cdot \omega)(f(p))(f_*v_1, \ldots, f_*v_k) \\ &= g(f(p))\omega(f(p))(f_*v_1, \ldots, f_*v_k) \\ &= (g \circ f)(p) \cdot f^*\omega(p)(f_*v_1, \ldots, f_*v_k) \end{aligned}$$

This is written in short as $f^*(g \cdot \omega) = (g \circ f) \cdot f^*\omega$.[1] You will prove some other such important formulae in the following exercises.

Exercise 2.3. Fix a smooth function $f : \mathbb{R}^n \to \mathbb{R}^m$. Recall that for each $a \in \mathbb{R}^d$, $dx_i \in \Omega^1(\mathbb{R}^d_a)$ is the smooth 1-form given by $dx_i(v_a) = (v_a)^i$, the i-th coordinate of the vector v_a. By way of unravelling definitions, prove that for any $p \in \mathbb{R}^n$ and any $v_p \in \mathbb{R}^n_p$. we have

$$[f^*dx_i](p)(v_p) = \sum_{j=1}^{n} \left(\frac{\partial f^i}{\partial x_j}\bigg|_p \right) dx_j(p)(v_p)$$

You should be able to do so in three or four lines! Dropping the p's and $v_p's$ because they are implicit gives us the succinct formula

$$f^*dx_i = \sum_{j=1}^{n} \frac{\partial f^i}{\partial x_j} dx_j$$

Notice also that this is the same as df when $m = 1$.

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[1] The dot is scalar multiplication

Exercise 2.4. Fix a smooth function $f : \mathbb{R}^n \to \mathbb{R}^m$. Let ω and η by k and l-forms on $\mathbb{R}^m$ respectively. Again, by way of unravelling definitions, prove that for any $p \in \mathbb{R}^n$ and any $v_1, \ldots, v_{k+l} \in \mathbb{R}^n_p$. we have

$$f^*(\omega \wedge \eta)(p) = f^*\omega(p) \wedge f^*\eta(p)$$

Again, you should be able to argue this in about 3-4 lines. This is written succinctly as $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$.

This is the final (and likely the most important) computational tool for computations, and should use the previously proven facts, plus that $f^*(\omega + \eta) = f^*\omega + f^*\eta$ if ω and η are both k-forms. [You may want to prove this for yourself, but this is probably easy.]

Exercise 2.5. Fix a smooth function $f : \mathbb{R}^n \to \mathbb{R}^m$. Let ω be a k-form on $\mathbb{R}^m$. We will prove by induction on k that the pullback commutes with the exterior derivative; that is: $f^*(d\omega) = df^*(\omega)$. To start, let $p \in \mathbb{R}^n$ and fix any $v_1, \ldots, v_k \in \mathbb{R}^n_p$.

1) Prove that this is true in the base case $k = 0$; that is ω is a 0-form, or a smooth function.

2) Suppose that we know that the result is true for k-forms. We want to show that this is true for $k+1$ forms as well. Prove that $f^*(d(\omega \wedge dx_i)) = d(f^*(\omega \wedge dx^i))$ for any k-form ω and any index i.

3) Conclude the proof of the theorem by writing any $(k+1)$-form as a linear combination of $k+1$-forms of the form in part 2).

3 Computations Involving Forms

We now have enough tools to manipulate differential forms to any extent we would like. Let us first demonstrate an example. Suppose that α is the 2-form on $\mathbb{R}^3$ given by $\alpha = 3x^2y dx \wedge dy + \sin(2xz) dy \wedge dz$. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $f(a,b) = (a+b, 2a-5b, ab)$. Then the pullback $f^*\alpha$ is supposed to be a 2-form on $\mathbb{R}^2$. Setting $g(x,y,z) = 3x^2y$ and $h(x,y,z) = \sin(2xz)$, we have that by using Exercise 2.4 and the discussion before it, we have that

$$f^*\alpha = (g \circ f) f^*(dx \wedge dy) + (h \circ f) f^*(dy \wedge dz) = (g \circ f)(f^*dx \wedge f^*dy) + (h \circ f)(f^*dy \wedge f^*dz)$$

Clearly, $g \circ f = 3(a+b)(2a-5b)$ and $h \circ f = \sin(2ab(a+b))$. Also, by Exercise 2.3,

$$\begin{aligned} f^*dx &= \frac{\partial f_1}{\partial a} da + \frac{\partial f_1}{\partial b} db = da + db \\ f^*dy &= \frac{\partial f_2}{\partial a} da + \frac{\partial f_2}{\partial b} db = 2da - 5db \\ f^*dz &= \frac{\partial f_3}{\partial a} da + \frac{\partial f_3}{\partial b} db = bda + adb \end{aligned}$$

Moving along,

$$\begin{aligned} f^*dx \wedge f^*dy &= (da + db) \wedge (2da - 5db) \\ &= da \wedge 2da + da \wedge (-5db) + db \wedge 2da + db \wedge (-5db) \\ &= -5(da \wedge db) + 2(db \wedge da) \\ &= -5(da \wedge db) - 2(da \wedge db) \\ &= -7(da \wedge db) \end{aligned}$$

and

$$\begin{aligned} f^*dy \wedge f^*dz &= (2da - 5db) \wedge (bda + adb) \\ &= 2da \wedge bda + 2da \wedge adb - 5db \wedge bda - 5db \wedge adb \\ &= 2a(da \wedge db) - 5b(db \wedge da) \\ &= 2a(da \wedge db) + 5b(da \wedge db) \\ &= (2a + 5b)(da \wedge db) \end{aligned}$$

Therefore

$$\begin{aligned} f^*\alpha &= -21(a+b)(2a-5b)(da \wedge db) + \sin(2ab(a+b))(2a+5b)(da \wedge db) \\ &= [-21(a+b)(2a-5b) + \sin(2ab(a+b))(2a+5b)] \cdot da \wedge db \end{aligned}$$

Exercise 3.1. This should be reminiscent of some computations we have done in multivariable calculus. Though these may seem routine, they are indispensable for computing integrals! Use the above example calculation to compute the pullbacks $f^*\alpha$ for the following:

1) α the 3-form on $\mathbb{R}^4$ given by $\alpha = (xyz)dx \wedge dy \wedge dz + (xyw)dx \wedge dy \wedge dw$ and $f : \mathbb{R}^3 \to \mathbb{R}^4$ given by $f(a, b, c) = (ab, bc, ac, abc)$. Given that $\dim \Omega^3(\mathbb{R}^3) = 1$, what does this tell you about $f^*\alpha$?

2) α the 2-form on $\mathbb{R}^2$ given by $\alpha = dx \wedge dy$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(r, \theta) = (r\cos\theta, r\sin\theta)$. Does this look familiar?

3) α the 3-form on $\mathbb{R}^2$ given by $\alpha = dx \wedge dy \wedge dz$ and $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(\rho, \theta, \varphi) = (\rho\sin\theta\cos\varphi, \rho\sin\theta\sin\varphi, \rho\cos\theta)$. Does this look familiar?

Okay those are interesting because they relate to multivariable calculus, but the similarities do not end there. Here is an example computation to get you started. Let $\alpha = 2x^2\cos(y)dx - 9e^{xy}dy$ be a 1-form on $\mathbb{R}^2$. We can calculate its exterior derivative by setting $f(x, y) = 2x^2\cos(y)$ and $g(x, y) = -9e^{xy}$. Recall that by definition,

$$d\alpha = df \wedge dx + dg \wedge dy$$

We readily compute df and dg:

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 4x\cos y dx - 2x^2 \sin y dy \\ dg &= \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = -9ye^{xy}dx - 9xe^{xy}dy \end{aligned}$$

Putting things together and reordering, we obtain

$$d\alpha = (2x^2\sin y - 9ye^{xy})dx \wedge dy$$

Exercise 3.2. Compute the exterior derivatives of the following k-forms.

1) $\alpha = Pdx + Qdy$.

2) $\alpha = Pdy - Qdx$. What do the two above equations remind you of?

3) Let θ be the angle function $\theta = \arctan(y/x)$, defined on $\mathbb{R}^2 - \{0\}$. Find $d\theta$ in terms of x, y, dx and dy.

In multivariable calculus, we briefly glance over the identities that for a function $f : \mathbb{R}^3 \to \mathbb{R}$, we have that the curl of the gradient is 0: $\nabla \times (\nabla f) = 0$ and that if $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field, then the divergence of the curl is 0: $\nabla \cdot (\nabla \times f) = 0$. To conclude this worksheet, you will prove these identities using differential forms.

Exercise 3.3. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Using the known formula for df, apply d one more time and use the fact that $d^2 = 0$ to conclude that $\text{curl}(\nabla f) = 0$

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field with components $F = (P, Q, R)$. By computing exterior derivatives for the two forms

$$\begin{aligned} \omega &= Pdx + Qdy + Rdz \\ \eta &= Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy \end{aligned}$$

use that $d^2 = 0$ to conclude that div(curl(f))=0.