Spring 2025 DRP HW 4: Group Homomorphisms

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1 Notes

We continue moving forwards with our study on groups. So far, we've been exploring the properties of groups themselves, and their intrinsic properties. In the previous homework, you saw how a map between groups can tell you about the structure of the domain (eg., its possible normal subgroups). In this problem set, we will study build on this theme in more generality. We will also do some review exercises in linear algebra because linear maps share many similarities to group homomorphisms. Questions will vary in difficulty. Some problems may also be duplicated in other texts. If this is the case, try to answer these without referring to the text. If you forget the statement of a theorem or formula, however, you are free to look up some results online.

2 Warmup: Properties of Linear Transformations

Recall that given (real) vector spaces V and W, a linear map between them is a function $f : V \to W$ satisfying $f(av) = af(v)$ for $a \in \mathbb{R}$, and $f(v_1 + v_2) = f(v_1) + f(v_2)$, where addition on the left side is done in V and addition on the right side is done in W.[1] Intuitively, this means that "f commutes with vector addition". You may also think of this as "preserving the vector space structure". The theme of preserving some structure will show up many many more times in your mathematical career; here is an example.

> **Exercise 2.1.** Let Γ_1, Γ_2 be graphs. An arbitrary function $f : \Gamma_1 \to \Gamma_2$ simply takes in a vertex from $v \in \Gamma_1$ and returns a vertex $f(v) \in \Gamma_2$. However this does not really use the graph-ness of the domain or codomain; it is really just a function of sets! What extra condition should you impose on f so that "f preserves the graph structure"?

Returning back to linear algebra, it turns out that the linearity condition is enough to impose severe restriction on the properties of the codomain and domain. In fact, it is *so* restrictive that there is in some sense nothing interesting left (this isn't completely true, but for our purposes, it is).

[1] This is called *linearity*, not surprisingly.

Exercise 2.2. Which of the following are linear maps? Prove your answer.

1) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 3x$

2) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 3x + 1$

3) $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = x + y + z$

2) $f : \mathbb{R}^4 \to \mathbb{R}^2$, $f(x, y, z, w) = x^2 + y^2 + z^2 + w^2$

Exercise 2.3. Using intuition from the previous exercise, show that a function $f : \mathbb{R}^n \to \mathbb{R}$ is linear if and only if $f(x_1, \dots, x_n) = a_1x_1 + \cdots + a_nx_n$ for some real numbers $a_1, \dots, a_n$. That is, the output needs to be a linear combination of the input.

Use the above to show that a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if each component of the output is a linear combination of the input variables. That is, if $f(x_1, \dots, x_n) = (y_1, \dots, y_m)$, then $y_i = a_{i1}x_1 + \cdots + a_{in}x_n$ for some real numbers a_{ij} and all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Use the above to conclude that $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map if and only if $f(x) = Ax$ for a $m \times n$ matrix A. [Hint: Set the entries of A to be the a_{ij} from above.]

This takes us full circle back to the MATH1554 days, where we linear maps are defined to be matrix multiplications. This exercise provides rigorous reasoning on *why* it is taught that way in introductory linear algebra classes. We conclude this warmup with one more important notion, of which we will see more of in the coming weeks.

Exercise 2.4. An *isomorphism* of vector spaces is a linear map $f : V \to W$ which is also bijective.[a] Show that if $\{b_1, \dots, b_n\}$ is a basis for V, then $\{f(b_1), \dots, f(b_n)\}$ is a basis for W [Hint: check spanning and linear independence. You will need to use linearity liberally.]. Therefore V and W have the same dimension!

[a] The word isomorphism is a noun; to use it as an adjective, one says that V and W are *isomorphic*.

3 Group Homomorphisms

The requirements of a group homomorphism $f : G \to H$ is a natural relaxation of the linearity requirements. Because there is no scalar multiplication, and the group multiplication need not be commutative, the linearity $f(ag + h) = af(g) + f(h)$ relaxes to become $f(gh) = f(g)f(h)$. This relaxation imposes less structure on the domain and codomain, but still contains a good amount of information. Firstly, verify these basic properties.

Exercise 3.1. Let $f : G \to H$ be a group homomorphism. Show that $f(e_G) = e_H$ and that $f(g^{-1}) = (f(g))^{-1}$.

We now turn to homomorphisms involving $\mathbb{Z}$ to gain more intuition.

Exercise 3.2. Let $f : \mathbb{Z} \to \mathbb{Z}$ be a homomorphism. If $f(1) = 4$, what is $f(35)$?

Let $f : \mathbb{Z}^2 \to \mathbb{Z}$ be a homomorphism. If $f(3, 2) = 6$ and $f(7, 5) = -2$, then what is $f(16, -4)$? In general, what is $f(x, y)$? [Hint: use some linear algebra!] Bonus: What is the image of f? The kernel?

Okay, so that was all about homomorphisms between infinite groups. What about maps between finite and infinite groups? To do so, we need a definition. An element $g \in G$ has *order* n if $g^n = \underbrace{g \cdot g \cdot \dots \cdot g}_{n} = e$, and $g^k \neq e$ for any integer $0 < k < n$. Complete the next routine exercise to check your understanding of the definition:

Exercise 3.3. Compute the orders of all elements in the groups $\mathbb{Z}/12\mathbb{Z}$, S_3 and D_4.

We will now see how order plays a role in group homomorphisms.

Exercise 3.4. Let $f : \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/12\mathbb{Z}$ be a group homomorphism. Show that it is possible that $f(1) = 2$? Show that it is not possible that $f(1) = 1$. [Hint: Exercise 3.1] Repeat this analysis for various outputs and find all of the possible numbers that $f(1)$ could be.

Exercise 3.5. Show that there is no homomorphism from $\mathbb{Z}/6\mathbb{Z}$ to $\mathbb{Z}/8\mathbb{Z}$ except for the trivial one (sending everything to 0) [Hint: the map will be determined by where $1 \in \mathbb{Z}/6\mathbb{Z}$ is sent. Show why it cannot be mapped to anything except 0.] Adapt this argument to show that there is no nontrivial homomorphisms $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}$. In terms of order of elements, what goes wrong here?

Are there nontrivial homomorphisms $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ though? How many of them are there?

Exercise 3.6. Using intuition from Exercise 3.5, prove that for a general homomorphism $f : G \to H$, if an element $g \in G$ has order n, then the order of the image $f(g)$ must be a divisor of n. Use this exercise to go back and furnish one-line solutions for Exercises 3.4 and 3.5.