Spring 2025 DRP HW 6: Bridge to Manifolds

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1 Notes

This will be the final problem set on multivariable analysis before we dive into integration over abstract manifolds. We will fill in some details about integration on open sets in $\mathbb{R}^n$, and get insight on this using the change of variables formula. To set ourselves up for the next unit in multilinear algebra and manifold, we will also have a few (somewhat scattered) warmup exercises. Questions will vary in difficulty. Some problems may also be duplicated in other texts. If this is the case, try to answer these without referring to the text. If you forget the statement of a theorem or formula, however, you are free to look up some results online.

2 Warmup 1: Some Linear Algebra

Recall that a real vector space (or $\mathbb{R}$-vector space) is a set of vectors V, endowed with vector addition and scalar multiplication satisfying a bunch of properties [look these up if not familiar]. Recall also that the dimension of a vector space is the size of any basis; this is a well defined number, and we will only consider the case of finite dimensional spaces. The prototypical examples are $V = \mathbb{R}^n$ and its subspaces, but there are some examples which are a little more obscure. Here is one important one.

Exercise 2.1 (Dual Spaces)**.** Let V be a real, n-dimensional vector space, and denote $\text{Hom}(V, \mathbb{R})$ to be the space of all linear maps from V to $\mathbb{R}$. Prove that $\text{Hom}(V, \mathbb{R})$ is a vector space. What is its dimension? [Hint: If a linear map $\varphi : V \to W$ is also bijective, and $\{v_1, \dots, v_n\}$ is a basis of V, then show that $\{\varphi(v_1), \dots, \varphi(v_n)\}$ is a basis of W.] This space is denoted V^*, and is called the *dual space* to V.

Exercise 2.2 (Dual Maps)**.** At some point in the previous exercise, you associated to a basis of V, say $\{v_1, \dots, v_n\} \subset V$ to a *dual basis* $\{v_1^*, \dots, v_n^*\} \subset V^*$. Let $\varphi : V \to W$ be a linear map. One can form a dual map $\varphi^* : W^* \to V^*$ which sends a linear map $A \in W^*$ to the linear map $(A \circ \varphi) \in V^*$.[a] Firstly, verify that φ^* is linear, meaning that it may be interpreted in terms of some matrix. Next, prove that if X is the matrix of φ with respect to a basis for V and W, then X^T is the matrix of φ^* with respect to the dual basis of V^* and W^*. This gives an alternate way of defining the matrix transpose.

[a] This is called the pullback, and it is used extensively in topology and geometry. The name comes from the idea that we are "pulling back" a linear map defined on W to a linear map defined on V, via the map T.

3 Warmup 2: Interesting Uses of Topology

This section is meant to expose you to a little more geometry and topology. In the first warmup, we will gain intuition about how topologists think, and practice using the words *continuity*, *topology* and *homeomorphism*. The second exercise is an application of topological spaces to answer a classical problem in number theory.

Exercise 3.1 (One-Point Compactifications)**.** Let (X, τ) be a topological space. We construct a new topological space (X^*, τ^*), where X^* is just X along with a single other point, which we will call ∞. τ^* (which, remember, are the open sets of X^*) will consist of everything τ itself along with all sets of the form $U = (X - C) \cup \{\infty\}$, where C is closed and compact in X. The space (X^*, τ^*) is called the *one-point compactification* of (X, τ).

1) Justify the namesake of the construction by proving that X^* is compact [Hint: Why are open sets containing ∞ defined to miss a compact set?]

2) Show that the one point compactifications of $\mathbb{R}$ and $\mathbb{R}^2$ are homeomorphic to S^1 and S^2, respectively. Here, the topologies on $\mathbb{R}, \mathbb{R}^2$ are the usual ones, and the topologies on S^1 and S^2 are the subspace topologies. Do not write out explicit formulas for this. Write out the maps in English, then use your stereographic projections to argue that your maps are bijective, continuous, and have continuous inverses.

3) Prove that the one point compactification of $\mathbb{N}$ is the space $\{0\} \cup \{1/n \mid n \in \mathbb{N}\}$, where both spaces have the discrete topology (ie., everything is open). [Hint: There should be a natural choice for the homeomorphism $f : \mathbb{N}^* \to \{0\} \cup \{1/n \mid n \in \mathbb{N}\}$]

And you now turn to a striking application of topology to number theories, where you will establish the infinitude of primes.

Theorem 3.2. *There are infinitely many prime numbers.*

We begin with the integers $\mathbb{Z}$, and define the topology τ as follows. $U \subseteq \mathbb{Z}$ is in τ if and only if for every $a \in U$, there is a number $m > 0$ so that entire arithmetic progression $\{\ldots, a - 2m, a - m, a, a + m, a + 2m, \ldots\}$ is contained in U. For example, the set $\{\ldots, -3, 1, 5, 9, \ldots\}$ is open in (X, τ). Prove that τ is indeed a topology.

You might notice that every open set (which is nonempty) must be infinite, since it needs to at least contain an infinite progression. You might also notice that a pure arithmetic progression, like $\{\ldots, -3, 1, 5, 9, \ldots\}$ is both open and closed [why?]. Now suppose towards a contradiction that there are only finitely many prime numbers p. Use the above two remarks to prove the following

Exercise 3.4 (How many prime numbers are there?). Verify that

$$\mathbb{Z} - \bigcup_{p \text{ prime}} \{\dots, -2p, -p, 0, p, 2p, \dots\} = \{-1, 1\}$$

using the fundamental theorem of arithmetic. Noting that the union in the above question is a finite union (since it is assumed that there are only finitely many primes), derive a contradiction and conclude that there infinitely many prime numbers. [Hint: A set $A \subset X$ is closed if and only if $X - A$ is open. Show firstly that the union of finitely many closed sets is closed].

□

4 Integration over Open Sets in $\mathbb{R}^n$

Let us now briefly explore abstract integration on open sets in $\mathbb{R}^n$. We will keep this section brief because one rarely uses this to perform any calculations (theoretical or not), since the tools of multivariate calculus or differential topology (such as the general Stokes' Theorem) are much more useful. This is why topology is better than analysis. Nevertheless, there are still some important points which are worth highlighting, so you will do this here. First of all, some more warmup from our previous section.

Exercise 4.1. Show that the x-axis sitting inside $\mathbb{R}^2$ has measure zero. Bonus: Show that every proper subspace of $\mathbb{R}^n$ has measure zero. [Hint: $\mathbb{Q}$ is countable, so we may list them like $\mathbb{Q} = \{q_1, q_2, \dots\}$. Place an appropriately sized ball at each q_i and argue that you've covered all of the x-axis.]

Let us first clear up some of the mess that Spivak made with his integration chapter. Let us first take a brief detour to our previous homework regarding partitions of unity. Since open sets are countably paracompact, we may apply Theorem 3.6. In particular, we took a locally finite refinement consisting of open balls, and defined functions on these balls. This shows that the functions from our partition of unity φ are actually compactly supported (since they have bounded support too, being contained in the union of finitely many balls). This allows us to rewrite the paragraph at the top of page 65 in Spivak.

Let $A \subseteq \mathbb{R}^n$ be an open set, and let $f : A \to \mathbb{R}$ be a locally bounded function (ie., for each $x \in A$, there is an open set U containing x for which f is bounded on U). Also suppose that the set of discontinuities $\{x \in A \mid f \text{ is discontinuous at } x\}$ has measure zero. Now choose an admissible open cover $\mathcal{O}$ of A and a partition of unity Φ subordinate to $\mathcal{O}$. We want to consider the integral

$$\int_A \varphi|f| = \int_{\operatorname{supp}\varphi} \varphi|f|$$

for every $\varphi \in \Phi$. Note firstly, that since $\operatorname{supp}\varphi$ is compact, it is certainly bounded, and thus is contained in a rectangle R.

Exercise 4.2. We shall prove that $\varphi|f|$ is integrable over $\operatorname{supp}\varphi \subseteq R$ using Theorem 3-8. In particular, we need that $\varphi|f|$ to be bounded over R, and $\varphi|f|\mathbb{1}_{\operatorname{supp}\varphi}$ to have discontinuities at only a set of measure zero.

1) Use compactness and local boundedness of f to prove that $\varphi|f|$ is bounded over R.

2) Note that $\varphi|f|\mathbb{1}_{\operatorname{supp}\varphi} = \varphi|f|$ on R. Then check that in general, if α and β are functions with a common domain, with α continuous, then the set of discontinuities of $\alpha\beta$ is contained in the set of discontinuities of β. [a] Conclude that $\int_A \varphi|f|$ exists and is finite.

[a] Multiplying by a continuous function can only fix discontinuities, not introduce them.

We are now ready to define the integral. Firstly,

Exercise 4.3. Now in the construction of the φ_α's, we started with some φ_i's and then reindexed. Therefore the φ_α's may be arranged in a sequence (well, the ones that aren't zero!). Carefully describe the construction of the integral $\int_A f$ from Spivak page 65.

There are two final things to prove, which are technical and stray away from our course, so we will only state them here in passing.

Theorem 4.4. *The definition of the integral (in the extended sense) is well-defined (ie., it did not depend on the choice of partition of unity, nor the choice of open cover). Moreover, when A and f are bounded, then this agrees with the integral we are all familiar with (ie., we can use Fubini's theorem, etc.)*

I shall conclude this problem set by remarking that this is *not* a useful definition, and no one in practice actually uses this to do anything. Many theories which require partitions of unity assert the existence of something, but rarely tells you how to calculate it. In our case, it simply provides a theoretical framework of intgration so that we may bootstrap our way up to integrating over manifolds. Indeed, most integral calculations in geometry/topology are done using Stokes theorem and its derivatives, and in analysis, the Lebesgue integral.