DRP 2024 HW 5: Partitions of Unity

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1 Notes

This is a miscellaneous collections of problems studying the beginning of manifold theory. We will also explore an important theoretical tool called the partition of unity. One usually interprets this as a tool to glue together local constructions to a global one. This is especially useful in manifold theory because manifolds are by nature local objects. Questions will vary in difficulty. Some problems will also be duplicated in Spivak's *Calculus on Manifolds*. If this is the case, try to answer these without referring to the text. If you forget the statement of a theorem or formula, however, you are free to look up some results online or from your own notes from my multivariable class.

2 Warmup: Bump Functions on $\mathbb{R}^n$

Though we have already done this in class, it is a good idea to revisit these ideas formally and write them up. We will use them in the next section (and then, bizarrely, never again). Recall that a function defined on Euclidean space is smooth if all of its partial derivatives of all orders exist at all points in the domain.

Exercise 2.1 (A Piecewise Smooth Function on $\mathbb{R}^n$)**.** Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

By way of elementary calculus, prove that f is smooth.

Exercise 2.2. Using the notation in Exercise 2.1, and taking two real numbers $a < b$ as parameters, prove that the function

$$g(x) = \frac{f(b-x)}{f(b-x) + f(x-a)}$$

is smooth on $\mathbb{R}$, equals one on $(-\infty, a]$, equals zero on $[b, \infty)$ and is in between zero and one on (a, b).

Exercise 2.3 (A Bump Function on $\mathbb{R}^n$)**.** Let $B_r(x)$ be the *open* ball of radius r centered at $x \in \mathbb{R}^n$. Using the previous exercise, construct a smooth function $h : \mathbb{R}^n \to \mathbb{R}$ so that on $\overline{B_a(0)}$, h is identically one, on $\mathbb{R}^n - B_b(0)$, h is zero, and $0 < h < 1$ everywhere else.

Exercise 2.4 (A Visual Application)**.** Consider the following interpolation problem. One has n *disjoint* discs in the plane, say $\{(x-h_i)^2+(y-k_i)^2 \leq r_i^2\}_{i=1}^n$, and n prescribed real numbers λ_i. Suppose also that there is a number $\epsilon > 0$ so small that the circles are disjoint even after extending the radius of each by ϵ. Construct a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ so that $f(x) = \lambda_i$ whenever x is in the i-th disc, $0 < f(x) < 1$ whenever x lies withing ϵ from a disc, and $f(x) = 0$ everywhere else. First construct a closed form solution for this problem, then graph it using any software of your choice and be ready to present it. What applications might this be useful for?

Note that these exercises were not chosen randomly. They are used in the following.

3 Partitions of Unity

We will begin with a little bit of technicals; this is necessary and helps strip our problem to what is *actually* relevant. Fear not, though, this is mainly preprocessing; all of this goes into only the first two lines of the whatever we will prove.

Definition 3.1. Let $\mathcal{O}_1 = \{U_\alpha\}_{\alpha \in A}$ and $\mathcal{O}_2 = \{V_\beta\}_{\beta \in B}$ be two open covers of a set $X \subseteq \mathbb{R}^n$. We say that $\mathcal{O}_2$ is a **refinement** of $\mathcal{O}_1$ if $\mathcal{O}_2$ still covers X, and every open set in $\mathcal{O}_2$ is contained in an open set of $\mathcal{O}_1$ (ie., if each V_β is contained in some U_α).

Definition 3.2. A **locally finite refinement** of an open cover $\mathcal{O} = \{U_\alpha\}_{\alpha \in A}$ of a set $X \subseteq \mathbb{R}^n$ is an refinement $\mathcal{O}' = \{V_\beta\}_{\beta \in B}$ so that each $x \in X$ has a neighborhood A which only intersects finitely many V_β (ie., $A \cap V_\beta \neq \varnothing$ for only finitely many $\beta \in B$). A **countably locally finite refinement** is a locally finite refinement which is countable (ie., in the notation above, this means the indexing set B is countable).

Definition 3.3. A set $X \subseteq \mathbb{R}^n$ is **countably paracompact** if every open cover admits a countably locally finite refinement.

Here is why what we just said is important. You certainly do not have to prove it, but it is good to know that while the main use of this section is to prove the existence of a partition of unity for manifolds, this is true for a larger class of sets.

Theorem 3.4. *Every manifold is countably paracompact.*

Now, partitions of unity come in several different forms, some weaker and some stronger depending on the assumption we put on the set. In our case, manifolds are often extremely well behaved (eg., being countably paracompact) which allows us to construct one of the strongest forms of PoU's, detailed below. Recall that the **support** of a function $f : X \to \mathbb{R}$ is the closure of the set over which f does not vanish: $\operatorname{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}}$.

Definition 3.5 (Partition of Unity)**.** Given a set $X \subseteq \mathbb{R}^n$ and an open cover $\mathcal{O} = \{U_\alpha\}_{\alpha \in A}$, a set of smooth function $\{\varphi_\alpha : X \to \mathbb{R}\}_{\alpha \in A}$ (indexed over the *same* set A) is called a **partition of unity subordinate to $\mathcal{O}$** if

I) $0 \le \varphi_\alpha \le 1$ for all $\alpha \in A$

II) $\text{supp}(\varphi_\alpha) \subseteq U_\alpha$ for every $\alpha \in A$

III) The collection of supports $\{\text{supp}(\varphi_\alpha)\}_{\alpha \in A}$ is locally finite. That is, every $x \in X$ has a neighborhood A which intersects only finitely many $\text{supp}(\varphi_\alpha)$. [Note that the original open cover is *not* required to be locally finite (and indeed in many cases is not).]

IV) For each x,

$$\sum_{\alpha \in A} \varphi_\alpha(x) = 1$$

Notice that from point III), the sum in IV) is a finite sum. This is important because it automatically curbs any potential issues about convergence or infinite sums. Now this is a technical definition, but we have actually already done a lot of the work! We just have to now glue things together in the correct order.[1]

Theorem 3.6 (Existence of PoU's)**.** *Every countably paracompact set $X \subseteq \mathbb{R}^n$ admits a partition of unity subordinate to any open cover.*

Proof. We will begin by reducing the general case to the case of a countable set of balls using countable paracompactness.

Exercise 3.7. Let $\mathcal{O} = \{U_\alpha\}_{\alpha \in A}$ be any open cover for X. Prove that there is a countable locally finite refinement $\mathcal{B} = \{B_{r_i}(x_i)\}_{i \ge 1}$ of $\mathcal{O}$ *consisting only of open balls* in $\mathbb{R}^n$.

This is the main step in the proof. The rest will now be to apply bump functions and make sure that the ideas are arranged in the correct order.

Exercise 3.8. For each ball $B_{r_i}(x_i)$ in our refinement, show that there exists a smooth function $g_i : \mathbb{R}^n \to \mathbb{R}$ so that $g_i = 1$ on $B_{q_i}(x_i)$ for some $q_i < r_i$ (take for example $q_i = r_i/2$ for concreteness), $g_i = 0$ outside of $B_{r_i}(x_i)$, and $0 < g_i < 1$ in $B_{r_i}(x_i) - B_{q_i}(x_i)$.

Having done this for every ball in the refinement, define now

$$g(x) = \sum_{i \ge 1} g_i(x)$$

[1] This is ironic because oftentimes it is the partition of unity that glues!

Exercise 3.9. The local finiteness of $\mathcal{B}$ shows that the sum is finite in some neighborhood of any $x \in X$. Argue that on a (potentially smaller) neighborhood of x, the terms that are nonzero in the sum are the same. More precisely, if $i_1, \ldots, i_k$ are so $g_{i_j}(x)$ are the only nonzero terms at x, then g_{i_j} are also the only nonzero terms in a some neighborhood of x.[a] Deduce that g is a smooth function. Finally, verify that $g(x)$ is *strictly* positive on X.

[a] This is a minor detail; it is possible that the set of g_i's which are nonzero can change in a neighborhood of x. The fact that these do not change takes care of this anomaly.

As the penultimate step, define

$$\varphi_i(x) = \frac{g_i(x)}{g(x)}$$

for each i. It is clear from construction that $0 \leq \varphi_i \leq 1$ and that $\sum_{i \geq 1} \varphi_i(x) = 1$ for all x [if this is not clear, write it out.] The final step is to re-index our functions in terms of α's. Recall that we are looking for a partition of unity subordinate to the original (possibly uncountable) cover $\mathcal{O}$, not the well-behaved one $\mathcal{B}$. We can relate them in the following natural way; since each $\mathcal{B}$ is a refinement of $\mathcal{O}$, every $B_{r_i}(x_i)$ is contained in some $U_\alpha \in \mathcal{O}$. This leads us to define, for $i \geq 1$, $\alpha(i)$ to be some $\alpha \in A$ for which $B_{r_i}(x_i) \subseteq U_\alpha$. Then define

$$\varphi_\alpha(x) = \sum_{i:\alpha(i)=\alpha} \varphi_i(x)$$

Exercise 3.10. The functions φ_α form our partition of unity. Verify that all of the conditions are satisfied [this should not be terribly difficult, since all we have done is reindex]. For criterion II), you may use the fact that

$$\overline{\bigcup U_\alpha} = \bigcup \overline{U_\alpha}$$

□