DRP 2024 HW 2: Group Actions

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1 Notes

This is a collection of problems covering Chapter 6 and 7 of Artin's *Algebra*. Each problem block is themed, usually with an insightful result at the end or is information I think is important to understanding group actions. Questions will vary in difficulty. Some problems will also be duplicated in Artin or other books. If this is the case, try to answer these without referring to the text.

Group actions are probably the way that most people think about groups subconsciously. After all, groups are simply an abstract collection of objects endowed with an operation. What we *really* care about is how they can tangibly act on relevant objects. For example, when we think of S_n as the permutation group on n elements, that's not what S_n *really* is - the permutation jargon is a formal way of saying that S_n acts on $\{1, \ldots, n\}$. For when one says that group theory is the study of symmetry, they do not mean symmetries of the *group itself*, but rather the objects they act on. As it turns out, studying the way groups act on objects actually sheds light on the groups themselves, and this problem set will look to exposit some of those insights. Your first exercise will be to watch this excellent video by Grant Sanderson. `https://www.youtube.com/watch?v=mH0oCDa74tE&ab_channel=3Blue1Brown`.

2 Groups Actions are Everywhere!

As I have claimed, group actions show up as often as groups do, and they reveal much about the extra structure of the group. For the following groups G and set X, find an a map $a : G \times X \to X$ satisfying the group action axioms. Note that the action is not given to you; you must define it yourself (they also may not be unique).

1. The group D_4 acting on the unit square.
2. The group of orthogonal 3×3 matrices acting on the space of lines in $\mathbb{R}^3$
3. Given a graph Γ, the automorphism group $\mathrm{Aut}(\Gamma)$ acting on Γ.
4. $\mathbb{Z}^n$ acting on $\mathbb{R}^n$.[1]
5. S_n acting on $\mathbb{R}^n$.

3 Groups Acting on Themselves

You have seen in the proof of Cayley's theorem that we can often study a group by studying how it either acts on itself or on the cosets of a subgroup. this section will be a compilation of some nice results obtained through considering a group acting on themselves or a set of cosets. As a warmup, recall this familiar exercise

[1] Compare to the covering maps at the end of the first problem set... is this a coincidence?

Exercise 3.1. *Let H be a subgroup of G of index two. Prove that H is normal in G*

Notice that for any even number, 2 is the smallest prime divisor. The following exercise is thus an extension of the previous one.

Exercise 3.2. *Let $|G| = n$, and let p be the smallest prime divisor of n. If H is a subgroup of index p, show that H is normal in G. [Hint: Show that H is the kernel of a permutation representation.]*

Recall that a group is *simple* if it has no proper normal subgroups.

Exercise 3.3. *Use the previous exercise to show that no group of order p^2 for any prime p can be simple.*

Exercise 3.4. *Classify all groups of order 6 by showing that if G is nonabelian, then it must have a non-normal subgroup of order 2.*

I will end this section by remarking that groups acting on cosets are an integral part of the proof of the Sylow theorems, a collection of combinatorial results regarding finite groups. I could have included the proofs as guided exercises here, I think there are more interesting things to say.

4 Quotients by a Group Action

A common theme in math is studying the symmetries of an object, and then *quotienting* by identifying everything that lives in the appropriate *symmetry bin.* We see this in the construction of the quotient group, where two elements of G are identified if they differ by an element of the normal subgroup G. Though the projection map $p : G \to G/H$ is complicated in general, it has a simple meaning: send an element g to the set of all elements which differ by an element of H. Thus, if the group G/H were significantly simpler than G, and there is not too much loss of information going from G to G/H, we can learn about G by studying G/H. Here is an example.

Exercise 4.1. *Let H be a normal subgroup of G, and let $\varphi : G \to K$ be a homomorphism. Find a condition on φ so that the map $\widetilde{\varphi} : G/H \to K$ given by $\widetilde{\varphi}([g]) = \varphi(g)$ is well defined. In the literature, we say that the following diagram commutes*

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & K \\ \downarrow{\scriptstyle p} & \nearrow_{\widetilde{\varphi}} & \\ G/H & & \end{array}$$

We will use this theme to examine the structure of quotients of familiar objects. Complete the following exercises by comparing to the final few problems from last month's homework:

Exercise 4.2. *Show that $\mathbb{Z}$ acts on $\mathbb{R}$, and $\mathbb{Z}^2$ acts on $\mathbb{R}^2$.*

Now given a group action G on a set X, we can consider the orbit space X/G, whose elements are the orbits (ie., $[x] \in X/G$ is the set $[x] = \{gx \mid g \in G\}$).

Exercise 4.3. *Describe the quotient spaces $\mathbb{R}/\mathbb{Z}$ and $\mathbb{R}^2/\mathbb{Z}^2$.*

Now, the real line, and the plane enjoy many more properties than just being groups. They are **topological spaces**, and more importantly, what we call **manifolds**. Though we will not study this too much more, hopefully you will be convinced that it is important to study functions on them; that is, functions from the circle to $\mathbb{R}$ or from the torus to $\mathbb{R}$. However, notice that it may be difficult to define the terms continuous/differentiable on these spaces in the usual way. Consider the circle, for example, which we might think of as living in $\mathbb{R}^2$. Recall that a function being continuous or differentiable requires us to take a limit of the form $\lim_{(x,y)\to(a,b)}$. However, this is not a suitable definition for continuity on the circle, because the function is defined only on the circle, not in the ambient space. Thus, the limit will not make sense, suggesting that we review our definitions.

Exercise 4.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Come up with conditions on f so that there is a function $\widetilde{f} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ satisfying $\widetilde{f}([x]) = f(x)$ for all $x \in \mathbb{R}$. Do the same for the torus.*

The point here is that to study the continuity of $\widetilde{f}$, it will suffice to study f instead, where we understand calculus much better. This line of reasoning whereby one take a foreign topological object and does calculus on it indirectly via a transport to $\mathbb{R}^n$ is called **calculus on smooth manifolds** and is an important part of my research. But there is much more! One way to interpret Exercise 5.4 is that 1-periodic functions on $\mathbb{R}$ induce functions on the circle. But we understand periodic functions very well! After all, the entire field of Fourier analysis studies periodicity. Hence there is an intimate relationship between Fourier Analysis and functions on the circle (perhaps a little bit of foreshadowing?).

5 Burnside's Lemma

Let G be a finite group acting on a finite set X. Given an element $g \in G$, let $X_g = \{x \in X \mid gx = x\}$ be the subset of X whose elements are fixed by g. Let $G_x = \mathrm{Stab}_G(x)$. To get a sense of how this works, consider a "true-false" table for the set $G \times x$ where you write a 1 in the position (g, x) if $gx = x$, and a zero otherwise. Do this for the group D_4 acting on the**corners** of a square. Also do this for the group S_3 acting on the set $\{a, b, c\}$.

Exercise 5.1. *Prove the formula*

$$\sum_{x \in X} |G_x| = \sum_{g \in G} |X_g| \tag{1}$$

*Can you interpret the formula using the truth-tables you made in the earlier discussion? Now use the formula to prove **Burnside's formula**:*

$$|G|\lambda = \sum_{g \in G} |X_g|$$

where λ is the number of distinct orbits in S.

We will now see a few applications of Burnside's formula to classical problems.

5.1 Colouring Necklaces

Consider a necklace made from n beads, each of which is to be coloured from a list of k colours. Let us say that two necklaces are equivalently coloured if they differ by some rigid transformation (ie., some reflection or rotation). We will be interested in the number of non-equivalent colourings.

Exercise 5.2. *Use Burnside's lemma to find a formula for the number of non-equivalent colourings of a n-bead necklace using k colours. Compute this number explicitly for a 6-bead necklace with 3 colours and a 7-bead necklace with four colours.*

By the way, I don't actually know the answer to this problem... I know that for prime n, there is a nice closed form but not sure about the general case.

5.2 Fermat's Little Theorem

In this subsection, we see a surprising application of group actions to prove Fermat's Little Theorem, a result in number theory stating that for an integer a and any prime number p, $a^p = a \pmod p$.

Let us start with a regular p-gon for a prime p and consider the subgroup H of rotations of D_p acting on the vertices of the p-gon. If we label the vertices using the numbers 1 through p, we may say that two labellings are rotationally equivalent if some element of H takes one labelling to the other.

Exercise 5.3. *Find a formula for the number of rotationally distinct labellings of the vertices of the p-gon. Use this to prove Fermat's Little Theorem.*