DRP 2024 HW 1: Elementary Analysis

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1 Notes

This is a collection of problems covering Chapters 1 and 2 of Abbott's *Understanding Analysis*. Each problem block is themed, usually with an insightful result at the end or will be presented in language relevant to topology. Questions will vary in difficulty. Some problems will also be duplicated in Abbott or other books. If this is the case, try to answer these without referring to the text.

2 Supremums and Infimums

Supremums and infimums are interesting (and necessary) because in the infinite, the functions max and min are generally not well-defined. Passing to the "greatest lower bound" instead of the "minimum", however, allows us to speak about most of what we care about when we want to find a minimum. Here are a few exercises to get comfortable with the definitions.

Exercise 2.1. *Let $A \subset \mathbb{R}$ have some finite infimum. Show that a real number $\alpha = \inf A$ if and only if for every $\epsilon > 0$, there is an element $x \in A$ so that $\alpha \leq x < \alpha + \epsilon$*

Exercise 2.2. *Recall that a ϵ-neighborhood of a point $x \in \mathbb{R}$ is the open interval $B_\epsilon(x) = (x-\epsilon, x+\epsilon)$. Let $A \subset \mathbb{R}$ have a finite supremum. Prove that a real number α is the supremum of A if and only if $\alpha \geq a$ for every element $a \in A$ and that for all $\epsilon > 0$, $B_\epsilon(x)$ contains an element of A.*

The above exercise is of special importance of topology. It states informally that we cannot separate the point α from A by open sets, but we can for any number strictly greater than α.

3 Cardinalities and Functions

This section does not contain any topological content but rather exposits one of my favourite proofs in elementary analysis. I do not know why this isn't taught more in college, but it should be. Recall that if $f : A \to B$ is an injective (1-1) function, then $|A| \leq |B|$. Thus, to establish that A and B have the same cardinality, it suffices to either establish a bijection between A and B, or equivalently, a pair of injections $A \to B$ and $B \to A$. In analysis one learns that $\mathbb{N}$ has the same cardinality as each of the subsets $2\mathbb{N}, 3\mathbb{N}, \ldots$ and perhaps even more surprisingly, as $\mathbb{Z}$. These can all be done by simple "pairing" arguments, but the first nontrivial example is that $|\mathbb{N}| = |\mathbb{Q}|$. This is usually done by some "snaking" Cantor-like argument, which can get very confusing. Follow these steps for an alternative.

Exercise 3.1. *Find an injection $f : \mathbb{N} \to \mathbb{Q}$.*

By the previous discussion, it will suffice to construct an injection $g : \mathbb{Q} \to \mathbb{N}$. To this end, recall that

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{N}, p, q \text{ coprime} \right\}$$

Let $\mathbb{Q}^+$ be the set of all fractions with $p > 0$.

Exercise 3.2. *Let $g : \mathbb{Q}^+ \to \mathbb{N}$ be given by*

$$g\left(\frac{p}{q}\right) = 2^p 3^q$$

Prove that g is injective [Hint: uniqueness of prime factorization]. Why did we insist that the domain is $\mathbb{Q}^+$ and not $\mathbb{Q}$?

Now, we have restricted our domain to the positive half, so to complete the argument, we need to account for the neglected half. There are many ways to do so, but here is the easiest way to do so.

Exercise 3.3. *Extend the previous function g to be defined on all of $\mathbb{Q}$ so that it is still injective. [Hint: uniqueness of prime factorization. How can you ensure that if $x < 0$, $g(x)$ attains a different value than if you had plugged in any positive number?]*

Alternatively, prove $|\mathbb{Q}^+| = |\mathbb{Q}|$ using any other method. In either case, conclude that $|\mathbb{Q}| = |\mathbb{N}|$.

4 A Topological View on Convergence

As a warmup, as well as practice with ϵ-type arguments, establish the following limits using the formal definitions.

Exercise 4.1. *Verify the following limits.*

1)

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

2)

$$\lim_{n\to\infty} \frac{n+1}{n} = 1$$

2)

$$\lim_{n\to\infty} \frac{n}{n+\sin n} = 1$$

Exercise 4.2. *Let the sequence x_n be defined with the recursive formula*

$$x_{n+1} = \frac{1}{4 - x_n}$$

*with $x_1 = 3$. Prove that x_n converges [Hint: do **not** use $\epsilon - N$!]. Now take limits on both sides of the equation compute $\lim_{n\to\infty} x_n$.*

The convergence of a sequence or series is of extreme importance in topology. There is a notion of "closure": a set is special if it contains all of its limit points. We will see this now.

Exercise 4.3. *Let $I = (0, 1)$. Find a convergent sequence $\{x_i\}_{i=1}^{\infty}$ so that $\lim_{i\to\infty} x_i \notin I$.*

Replace I with the closed interval $J = [0, 1]$. Show that any convergent sequence in J has its limit point also in J.

Following the idea of a set containing all its limit points, we give, to all sets with this property, the provocative name "closed". As you have proven, all closed intervals are closed. However, not every closed set is a closed interval (say, for example $[0,1] \cup [3,4]$ or the ray $[0,\infty)$). There is however an intimate connection between a closed set and its complement.

Exercise 4.4. *Let $A \subset \mathbb{R}$ be closed and nonempty. Prove that $\mathbb{R} - A$ is not closed. [Hint: establish this intermediate claim. For any set $X \subset \mathbb{R}$, if all points $x \in X$ has a neighborhood $B_\epsilon(x)$ contained in X, then X cannot be closed.]*

In fact, more is true; the complement is actually open, but we haven't defined what it means to be open, though we will in the next chapter...

5 An Unconventional Proof of the AST

The Alternating Series test from calculus class states that if $\{a_n\}_{n=1}^{\infty}$ is a decreasing sequence such that $\lim_{n\to\infty} a_n = 0$, then the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n$$

necessarily converges. To prove this, we will take an unnecessary detour to *Dirichlet's* test and realize the AST as a corollary. To do this, establish this technical lemma, which is just a discrete version of integration by parts.

Exercise 5.1. *Let $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ be two sequences, with $s_n = x_1 + \ldots x_n$ denoting the partial sums. Verify the formula*

$$\sum_{j=m+1}^{n} x_j y_j = (s_n y_{n-1} - s_m y_{m-1}) + \sum_{j=m+1}^{n} s_j (y_j - y_{j-1}) \tag{1}$$

[Hint: How might one express x_i in terms of the s_j's?]

We are ready to lay the hypotheses for the Dirichlet test. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence for which its partial sums s_n are bounded (but not necessarily convergent), and $\{y_n\}_{n=1}^{\infty}$ a positive, decreasing sequence converging to zero. Dirichlet's test states that the sum

$$\sum_{n=1}^{\infty} x_n y_n$$

converges.

Exercise 5.2. *Let $M > 0$ be an upper bound for the partial sums s_n. Use Equation 1 to prove the bound*

$$\left|\sum_{j=m+1}^{n} x_j y_j\right| \leq 2M|y_{m+1}|$$

Use the above inequality to prove Dirichlet's test.

Exercise 5.3. *Prove the Alternating Series test using the Dirichlet test.*

6 Riemann Rearrangements

Recall that a series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if the series $\sum_{n=1}^{\infty} |a_n|$ converges, and *converges conditionally* if the first series converges and the second diverges. An example is the harmonic series, as well as many series that satisfy the hypotheses of the AST. The Riemann rearrangement theorem states that if one has a conditionally convergent series, then one can rearrange the terms of the series so that the convergent sum can be any arbitrary real number. Let us take a step back and think about this externally, because it sheds insight on why abstract mathematics is needed. For most of our mathematical lives, addition is a commutative operation (changing the order doesn't matter). However, in the infinite, this is far from the truth. Analysis as a mathematical discipline formalizes these rules that we take for granted, and set boundaries for when we can and can't perform such operations. As other examples, when are we allowed to swap the order of two limits? Or a limit and a derivative? Or a derivative and an integral? When is $\lim_{x\to\infty} \int_{-x}^{x} f(x)dx$ equal to $\lim_{x\to\infty} \lim_{y\to\infty} \int_{-y}^{x} f(x)dx$? Such operations are necessary in every STEM field, and you yourself have probably used these concepts without thinking. However, the reason that we are (usually) justified in doing these things are because of some vital analytical calculations. We shall explore one of them here.

Let $\sum_{i=1}^{\infty} a_i$ be a conditionally convergent series. Define two series (p_j for positive, n_j for negative) as follows: $p_j = a_j$ if $a_j \geq 0$, and $n_j = a_j$ if $a_j < 0$. That is, the p_j consists of all the positive terms and n_j consists of all the negative terms. For example, for the alternating harmonic series

$$\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

we have

$$\sum_{i=1}^{\infty} p_i = 1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \qquad \text{and} \qquad \sum_{i=1}^{\infty} n_i = -\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \dots$$

Exercise 6.1. *Given $\{a_i\}_{i=1}^{\infty}$ an arbitrary conditionally convergent sequence, prove that both $\sum_{i=1}^{\infty} p_i$ and $\sum_{i=1}^{\infty} n_i$ diverge.*

Exercise 6.2. *Let x be any real number. Prove that there is a rearrangement of the a_i's so that their sum converges to x using the following algorithm. Suppose firstly that $x > 0$. Add p_i's until the partial sum exceeds x, then add n_i's until the partial sum is lower than x. Then add p_i's again, and iteratively do this, oscillating around x to infinity. Fill in all the details carefully (ie., verify that this is actually possible, that it does indeed converge and that this works for all real numbers x).*

There is an interesting application of this theorem. For the following exercise, we will be using the alternating harmonic series.

Exercise 6.3. *Using the above algorithm, write a computer program to compute better and better approximations of $\pi/4$. For example, the first approximation might be*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \approx 0.783 \approx \frac{\pi}{4}$$

I do not know if computers actually employ similar approximation techniques as I came up with this problem during a scooter ride, but should be interesting nonetheless.